

## RESEARCH ARTICLE

**Bond diluted Levy spin-glass model and a new finite size scaling method to determine a phase transition**L. Leuzzi<sup>a,\*</sup>, G. Parisi<sup>a,b,c</sup>, F. Ricci-Tersenghi<sup>a,b,c</sup> and J.J. Ruiz-Lorenzo<sup>d</sup><sup>a</sup>*IPCF-CNR, UOS Roma, Piazzale A. Moro 2, 00185, Rome, Italy.*<sup>b</sup>*Department of Physics, University "Sapienza", Piazzale A. Moro 2, 00185, Rome, Italy.*<sup>c</sup>*INFN, Piazzale A. Moro 2, 00185, Rome, Italy.*<sup>d</sup>*Departamento de Física, Univ. Extremadura, Badajoz, E-06071 and BIFI, Spain.**(Received 00 Month 200x; final version received 00 Month 200x)*

A spin-glass transition occurs both in and out of the limit of validity of mean-field theory on a diluted one dimensional chain of Ising spins where exchange bonds occur with a probability decaying as the inverse power of the distance. Varying the power in this long-range model corresponds, in a one-to-one relationship, to change the dimension in spin-glass short-range models. Using different finite size scaling methods evidence for a spin-glass transition is found also for systems whose equivalent dimension is below the upper critical dimension at zero magnetic field. The application of a new method is discussed, that can be exported to systems in a magnetic field.

**1. Introduction**

Long-range (LR) models are such that their lower critical dimension is lower than the one of the corresponding short-range (SR) model. In particular, one can have a phase transition even in one dimensional systems, provided the range of interaction is large enough. One dimensional models with power-law decaying interactions actually allow to explore both LR and SR regimes by changing the power and enable to compare the ordered phase in and out of the range of validity of the mean field approximation. This is very useful for spin glass models that are known to have a rather complex states structure in the low-temperature phase in mean-field theory [1, 2]. Whether this structure exists in finite-dimensional models with short-range interactions is, though, still a matter of debate. Theories alternative to the mean-field one have been proposed [3, 4], but SR systems are very tough to study analytically. Numerical simulations have been, thus, extensively employed, though with no conclusive indication on the nature of the spin-glass (SG) phase in finite dimension nor on the existence of a thermodynamic SG phase in presence of an external magnetic field.

A one-dimensional spin-glass model with power-law decaying interactions [5–9] makes probes on larger linear sizes possible. Moreover, if bonds are diluted [10–13] the run time in numerical simulations grows simply as the size  $L$  of the system, rather than proportionally to  $L^2$ , as in fully connected systems. This is a fundamental issue because finite volume effects are strong in these models, so much that the very existence of the transition can be rather difficult to establish with canonical methods, e.g., when the interaction has a rapid decay and/or an

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external magnetic field is applied.

## 2. The 1D Levy Spin-Glass Model

The model investigated is a one dimensional chain of  $L$  Ising spins ( $\sigma_i = \pm 1$ ) and Hamiltonian

$$\mathcal{H} = - \sum_{ij} J_{ij} \sigma_i \sigma_j \quad (1)$$

The quenched random couplings  $J_{ij}$  are independent and identically distributed random variables taking a non zero value with a probability decaying with the distance between spins  $\sigma_i$  and  $\sigma_j$ ,  $r_{ij} = |i - j| \bmod (L/2)$ , as

$$\mathbf{P}[J_{ij} \neq 0] \propto r_{ij}^{-\rho} \quad \text{for } r_{ij} \gg 1. \quad (2)$$

Non-zero couplings take value  $\pm 1$  with equal probability. We use periodic boundary conditions and a  $z = 6$  average coordination number. Links are generated by repeating  $zL/2$  times the following process: choose randomly 2 spins at distance  $r$  with probability  $r^{-\rho} / \sum_{i=1}^{L/2} i^{-\rho}$ ; if they are already connected, repeat the process, otherwise connect them.

As the power  $\rho$  varies this model is known to display different statistical mechanics behaviors [5–7]. For the diluted case [10] they are reported in Tab. 1. Equivalence between one-dimensional systems with power-law decaying interactions and  $D$ -dimensional systems with short-range (nearest neighbor) interactions can be approximately written as [12]:

$$\rho - 1 = \frac{2 - \eta}{D} \quad (3)$$

where  $\eta$  is the critical exponent of the space correlation function for the short-range model. The relationship is exact at the mean-field threshold  $\rho = 4/3$  ( $D = 6$ ,  $\eta = 0$ ) and approximated below. The analogue of a 3D spin-glass in zero magnetic field (with  $-0.384(9) < \eta < -0.337(15)$  [14–18]) would then be a system with  $\rho \simeq 1.8$ . We will focus on this value in the present manuscript to introduce and test our method. Other model cases have been considered in Refs. [10, 11].

$\rho$	$D(\rho)$	transition type
$\leq 1$	$\infty$	Bethe lattice like
$]1 : 4/3[$	$[6 : \infty[$	2 <sup>nd</sup> order, MF
$]4/3 : 2[$	$[2.5 : 6[$	2 <sup>nd</sup> order, non-MF
2	2.5	Kosterlitz-Thouless or $T = 0$ -like
$> 2$	$< 2.5$	none

Table 1. From infinite range to short range behavior of the SG model defined in Eqs.(1,2).

## 3. Numerical simulations

We simulate two replicas  $\sigma_i^{(1,2)}$  using the parallel tempering algorithm (PT) [19]. The simulated sizes are  $L = 2^\kappa$ , with  $\kappa = 6, 8, 10, 12$ . The interval between

temperatures in the PT evolution is  $\Delta T = 0.05$ . The number of samples is  $N_J = 6400 - 90000$  depending on the size. All data used for our analysis are thermalized. Thermalization has been checked by measuring all observables on exponentially growing time windows until the last two points coincide within the statistical error.

#### 4. Critical point

The key observable to approach the critical behavior in disordered systems is the four-spins correlation function:

$$C_4(x) = \frac{1}{L} \sum_{i=1}^L \overline{\langle \sigma_i^{(1)} \sigma_i^{(2)} \sigma_{i+x}^{(1)} \sigma_{i+x}^{(2)} \rangle} \quad (4)$$

and its Fourier transform  $\tilde{C}_4(k)$ . In order to determine the critical point, a correlation length-like observable is usually defined on the 1D lattice as [9, 20]

$$\xi = \frac{1}{2 \sin k_1/2} \left[ \frac{\tilde{C}_4(0)}{\tilde{C}_4(k_1)} - 1 \right]^{1/(\rho-1)} \quad (5)$$

with  $k_1 = 2\pi/L$ . In Fig. 1 we present the  $\xi/L$  curves whose crossing point should tend, as  $L \rightarrow \infty$ , to  $T_c$ . In the inset we also show the behavior of

$$\chi_{SG} = L\tilde{C}_4(0), \quad (6)$$

another finite size scaling (FSS) function for the present model (in which  $\eta = \eta_{MF} = \rho - 1$  also for  $\rho > \rho_{MF}$ ). Due to the statistical error it is not straightforward to identify clear crossing points for  $\xi/L$ . Moreover, in both above mentioned cases, to extrapolate a clear limit of  $T_c$  as  $L \rightarrow \infty$  with a FSS interpolating function like  $a + bL^{-c}$ , cf. Fig. 4, three degrees of freedom are not enough and the interpolations are thus just indicative (see the following).

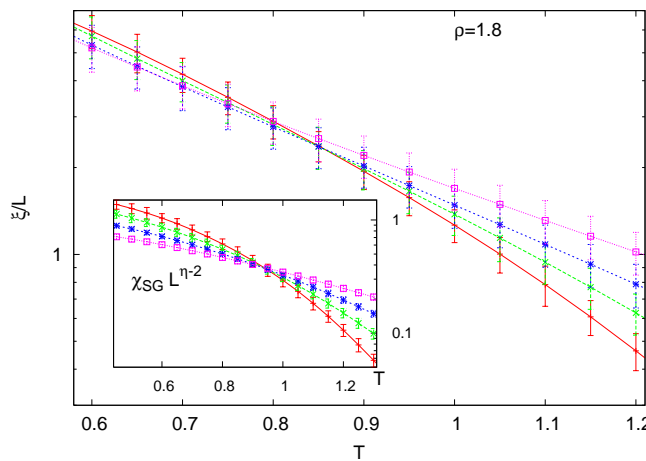


Figure 1. FSS function  $\xi/L$  vs.  $T$  for different simulated sizes at  $\rho = 1.8$  for  $L = 2^{6,8,10,12}$ . Inset:  $\chi_{SG} L^{\eta-2}$  vs.  $T$  for the same sizes.

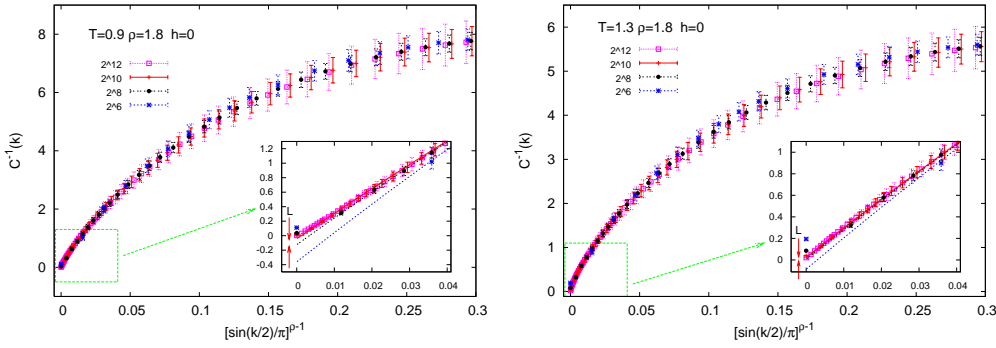


Figure 2. Average  $\tilde{C}_4^{-1}(k)$  vs.  $[\sin(k/2)/\pi]^{0.8}$  for systems of size  $L = 2^{6,8,10,12}$ . Left:  $T = 0.9$ , slightly below the critical region. Right:  $T = 1.3 > T_c$ . Insets: detail for low  $k$  (points) and comparison of the values in  $k = 0$  with the  $k \rightarrow 0$  limit of the interpolation  $A + B[\sin(k/2)/\pi]^{0.8}$  (lines).

## 5. A novel method

We can, otherwise, use the whole information provided by the  $\tilde{C}_4(k)$ . In Fig. 2 we plot  $1/\tilde{C}_4(k)$  vs.  $[\sin(k/2)/\pi]^{\rho-1}$  for  $\rho = 1.8$  for all simulated sizes both at temperature above  $T_c$  (right) and at a  $T \simeq T_c$  (left). We observe that finite size effects act in opposite ways on the value of  $\tilde{C}_4(0)$  (i.e.,  $\chi_{SG}$ ) and on the rest of the function  $\tilde{C}_4(k > 0)$  (see insets of Fig. 2): while  $1/\tilde{C}_4(0)$  tends to its thermodynamic limit from above,  $1/\tilde{C}_4(k > 0)$  and its interpolation at  $k = 0$  tend to the thermodynamic limit from below. Even though  $\tilde{C}_4(0)$  and  $\lim_{k \rightarrow 0} \tilde{C}_4(k)$  are the same object for  $L \rightarrow \infty$  their FSS scaling is qualitatively different.

Interpolating  $1/\tilde{C}_4(k)$  for small  $k$  at a given size and temperature as

$$F^{\text{fit}}(k) = A(L, T) + B(L, T)[\sin(k/2)/\pi]^{0.8} \quad (7)$$

we can analyze the  $L$  and  $T$  dependence of

$$A(L, T) \equiv \lim_{k \rightarrow 0} \tilde{C}_4(k) \quad (8)$$

and determine the transition from the FSS analysis of the points at which  $A(L, T) = 0$ , rather than using the FSS of the crossing points of Eq. (5) (or  $\tilde{C}_4(0)L^{\eta-1}$ ). For  $\rho = 1.8$ , the behavior of  $A$  in  $L$  and  $T$  is plotted in Fig. 3.

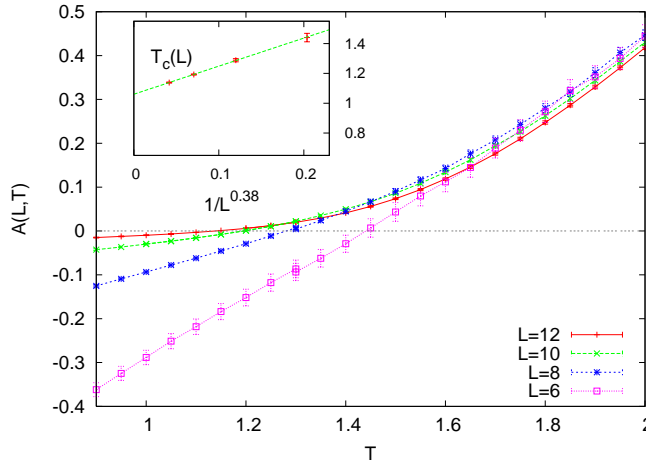


Figure 3. Behavior of  $A(L, T)$ , cf. Eq. (8), for  $L = 2^{6,8,10,12}$ . The points at which  $A = 0$  are finite size estimates of  $T_c$ .

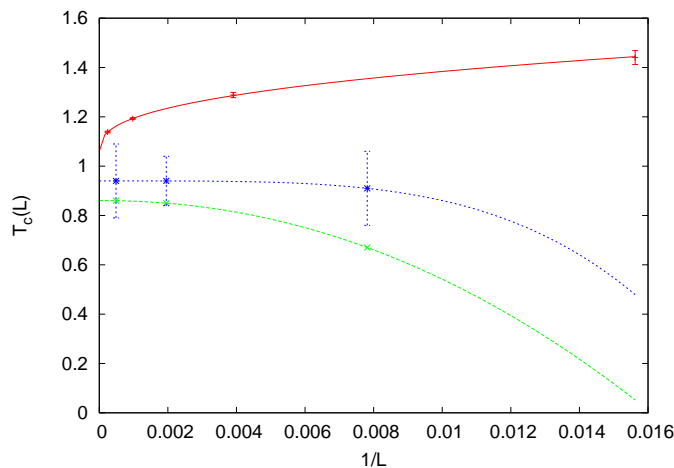


Figure 4. Comparison between the finite size estimates of  $T_c(L)$  by means of the canonical “crossings” methods (of the FSS functions  $\xi/L$  and  $\chi_{SG} L^{\eta-2}$ ) and of the  $A(L, T) = 0$  method, cf. Eq. (8). The fit for  $T_c$  from  $A(L, T) = 0$  has a  $\chi^2 = 0.075$  (full/red curve), while the other two interpolating curves (dashed/green for  $\xi/L$ , dotted/blues for  $\chi_{SG} L^{\eta-2}$ ) are only indicative (not enough d.o.f. to provide an estimate of statistical errors, no error bars defined on the  $T_c(L)$  from  $\xi/L$  crossings).

This method has the advantage of using high temperature data and one only needs to simulate systems down to the candidate  $T_c$ . As  $A(\infty, T)$  becomes negative, indeed, the functional form of the propagator in the paramagnetic phase breaks down and this provides evidence for a phase transition. In Fig. 4 we plot the finite size values of  $T_c$  obtained by this method and we compare them with the estimates derived from FSS functions  $\xi$  and  $\chi_{SG}$ . At  $\rho = 1.8$  we find  $T_c = 1.060(7)$  and  $1/\nu = 0.38(2)$  with a  $\chi^2 = 0.075$  on the same data of the previous analysis (same statistics, same thermalization times, same sizes).

We stress that also the definition of  $\xi$  as a correlation length in Eq. (5) is valid only in the paramagnetic phase and that below  $T_c$  this is just a scaling function without physical meaning. In that approach, though, in order to appreciate crossings of  $\xi_L(T)/L$  curves at different  $L$  one has to simulate the system also at temperatures below  $T_c$ , where thermalization times increase, and for at least five different sizes in order to provide enough points for a well defined FSS interpolation.

## 6. Correlation length estimate

We are interested in characterizing the above mentioned critical behavior by means of a growing correlation length, as it happens in ordinary continuous phase transitions in systems without quenched disorder. Also for this analysis our starting point is the four-spins correlation function, this time in position space. As one can notice from Fig. 5, we can identify two different decays of the correlation at a given  $T > T_c$ , if we are able to study long enough Levy-glass chains. We observe a slower power-law decay as  $x \ll L$ ,  $x^{-\alpha}$ , and a faster decay as  $x \sim L/2$ . Contrarily to what happens in short-range models at  $D \geq 2.5$  [14, 21], this second decay is also power-law (with power equal to  $\rho$ ) because the interaction correlation decays - by construction - as  $x^{-\rho}$  and, therefore, the  $C_4(x)$  cannot decay any faster.

From the definition of  $\eta$  as exponent of the anomalous decay of the correlation function as  $x^{-d+2-\eta}$ , with  $\eta = 3 - \rho$ , we obtain the relation  $\alpha + \rho = d + 1 = 2$  [22]. We can, thus, think about interpolating the whole  $C_4(x)$  behavior above the

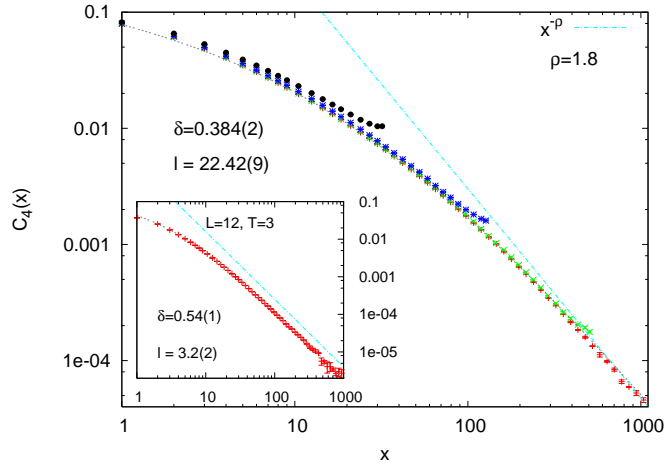


Figure 5. Four spins correlation function at high temperature ( $T = 2 \simeq 1.9T_c$ ) in log-log scale for all simulated sizes  $L = 2^{6,8,10,12}$ . Dotted lines: interpolation by means of Eq. (9). Dashed-dotted line:  $x^{-\rho}$ . No finite size effects are present, apart from the values at  $x \simeq L/2$ , and a crossover from a power law with exponent  $\alpha \simeq 0.2$  to a power law with  $\rho = 1.8$  can be identified for  $L > 10$ . Inset:  $C_4(x)$  for  $L = 2^{12}$  at  $T = 3$  compared to  $x^{-1.8}$ .

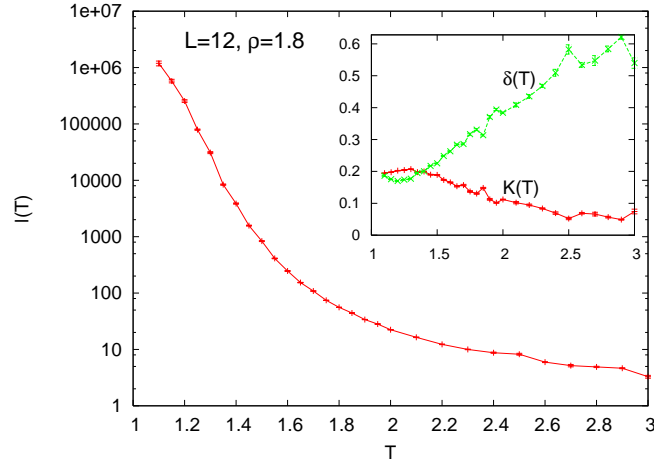


Figure 6. Correlation length parameter  $\ell$  of interpolating function  $C_4^{\text{fit}}(x)$  vs.  $T$  in log scale for  $L = 2^{12}$ . Inset:  $T$  behavior of fit parameters  $K$  and  $\delta$ .

critical point as

$$C_4^{\text{fit}}(x) \equiv K x^{-\alpha} \left[ 1 + \left( \frac{x}{\ell} \right)^{2\delta(\rho-\alpha)} \right]^{-1/\delta} \quad (9)$$

with  $\rho = 1.8$  and  $\alpha = 2 - \rho = 0.2$ .

From this interpolation we can look at the temperature behavior of the length-like parameter  $\ell$ , that is an estimate of the correlation length of the system, as far as the second power-law decay is observable ( $T > T_c$ ). In Fig. 6 we display the behavior of  $\ell(T)$  (and  $K$  and  $\delta$  in the inset) as  $T$  is lowered down to the estimated  $T_c$ . Fits with Eq. (9) are reasonable down to  $T = 1.1$ .

As temperature approaches the critical value from above, the simulated systems are too small to appreciate the existence of a crossover length  $\ell \gg L$  and, thus, the  $C_4(x)$  decay appears as a single power law. The same behavior remains below  $T_c$ , as shown in Fig. 7. This is incompatible with the onset of a plateau at any  $x$  whereas it is consistent with the clustering properties of the mean-field theory for spin-glasses.

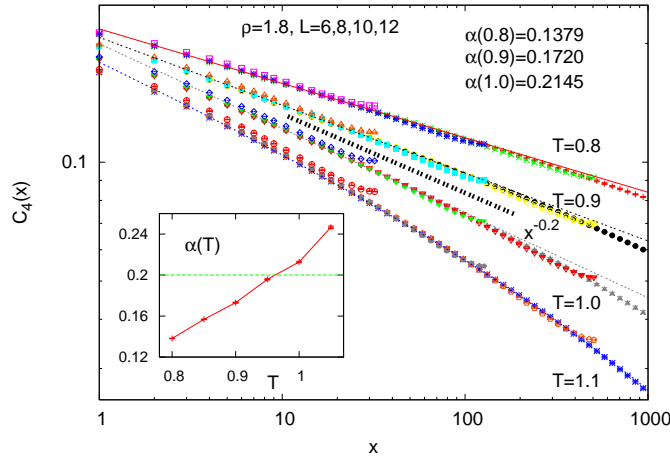


Figure 7. Four spins correlation function across the critical region ( $T = 0.8, 0.9, 1, 1.1$ ) in log-log scale for all simulated sizes  $L = 2^{6,8,10,12}$  and relative interpolating functions. At all  $T$ , finite size effects are seemingly small. Dotted lines: interpolation by means of Eq. (9) at  $T = 1.1$ ; for lower  $T$  the fit function is  $Ax^{-\alpha}$  is used and no crossover to a power law with  $\rho = 1.8$  is observed for any  $L$ , in agreement with  $\ell \gg L$ . The  $T$ -dependence of the value of  $\alpha$  is shown in the inset.

## 7. Conclusions

We have introduced a new FSS method to determine, by numerical simulations, the existence of a critical point in finite dimensional systems. Such method employs high temperature data. Thus, it requires lower thermalization times and disordered sample statistics with respect to canonical methods. It works well if one has a sufficient amount of points of the four-spins correlation function at low wavelength numbers  $k$ , that is, if the interaction range is not too broad (i.e., preferably out of mean-field) and linear size is long enough.

We have tested the method in the case of a bond-diluted 1D Levy spin-glass with a power-law decaying interaction outside the limit of validity of mean-field approximation and equivalent to a 3D nearest-neighbor interacting system on a cubic lattice. We have compared the results with those obtained by canonical FSS analysis on the same set of data and shown that they are compatible, cf. Fig. 4.

In position space, at  $T > T_c$  we have identified a crossover in the (power-law) relaxation decay of the four-spins correlation function from slow to fast (i.e.,  $x^{-\rho}$ ) relaxation, cf. Eq. (9). The crossover takes place at a correlation length that grows exponentially as  $T$  decreases. At  $T < T_c$  the decay is well interpolated by a simple power-law, providing no evidence for the plateau predicted by droplet [3] and TNT theories [4].

## 8. Acknowledgments

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- [22] This can be seen, e.g., in Ref. [10] for  $\rho = 1.5$ . In the inset of figure 6  $\alpha$  was erroneously misplaced with  $\delta$ .